1. Let f be a function with domain $D \subset \mathbb{R}^p$ and range in $\mathbb{R}^q$.
   (a) Define what it means to say that f is a bounded function.
   (b) If f is bounded, define the supremum norm of f. Check that (i) $\|f\|_D \geq 0$, and $\|f\|_D = 0$ if and only if $f(x) = 0\ \forall x \in D$; (ii) $\|cf\|_D = |c|\|f\|_D$; (iii) $\|f+g\|_D \leq \|f\|_D + \|g\|_D\ \forall f, g \in B_{pq}(D)$, which implies that $f \to \|f\|_D$ is a **norm** on $B_{pq}(D)$.

2. Let $\{f_n\}$ be a sequence of functions with domain $D \subset \mathbb{R}^p$ and range in $\mathbb{R}^q$.
   (a) Define what it means to say that $\{f_n\}$ is **pointwise** convergent on D.
   (b) Define what it means to say that $\{f_n\}$ is **uniformly** convergent on D.
   (c) Define what it means to say that $\{f_n\}$ is **Not uniformly** convergent on D.

3. (a) For each $n \in \mathbb{N}$, let f_n be defined for $x > 0$ by $f_n(x) = \dfrac{1}{nx}$. For what values of f does $\lim\limits_{n\to\infty} f_n(x)$ exist? Is the convergence uniform on the entire set of convergence? Is the convergence uniform for $x \geq 1$?
   (b) Show that, if we define f_n on $\mathbb{R}$ by $f_n(x) = \dfrac{nx}{1+n^2x^2}$, then $\lim\limits_{n\to\infty} f_n(x)$ exists for all $x \in \mathbb{R}$. Is this convergence uniform on $\mathbb{R}$?
   (c) Show that $\lim\limits_{n\to\infty} (\cos \pi x)^{2n}$ exists for all $x \in \mathbb{R}$. Is the limiting function contiuous? Is this convergence uniform?
   (d) Let f_n be defined on the interval $I = [0, 1]$ by the formula

$$f_n(x) = \begin{cases} 1 - nx, & \text{if } 0 \leq x \leq 1/n \\ 0 & \text{if } 1/n < x \leq 1. \end{cases}$$

   Show that $\lim\limits_{n\to\infty} f_n(x)$ exists on I. Is this convergence uniform on I?
   (e) Let f_n be defined on the interval $I = [0, 1]$ by the formula

$$f_n(x) = \begin{cases} nx, & \text{if } 0 \leq x \leq 1/n \\ n(1-x)/(n-1) & \text{if } 1/n < x \leq 1. \end{cases}$$

   Show that $\lim\limits_{n\to\infty} f_n(x)$ exists on I. Is this convergence uniform on I? Is the convergence uniform on $[c, 1]$ for $c > 0$?

4. Let f be a function with domain $D \subset \mathbb{R}^p$ and range in $\mathbb{R}^q$.
   (a) Define what it means to say that f is a **Lipschitz** function.
   (b) Define what it means to say that f is a continuous function on D.
   (c) Is a Lipschitz function always continuous?
   (d) Is a linear function $f : \mathbb{R}^p \to \mathbb{R}^q$ always Lipschitz? [May write $f(x) = Ax + b,$, where A is a $q \times p$ constant matrix, and b is a $q \times 1$ vector]
   (e) Is a Lipschitz function always differentiable?
   (f) Let f be a differentiable function defined on an open subset U of $\mathbb{R}^p$. Is f Lipschtz on U? Is f Lipschitz on any compact subset K of U?

5. Let $X = \{x_{mn}\}$ be a double sequence in $\mathbb{R}^p$.
   (a) Define what it means to say that x is a limit of $\{x_{mn}\}$.

(b) For each $m \in \mathbb{N}$, let $Y_m = \{x_{mn}\}$ be a sequence in $\mathbb{R}^p$ which converges to y_m. Define what it means to say that $\{Y_m \mid m \in \mathbb{N}\}$ are uniformly convergent.

6. Let $f : [a, b] \to \mathbb{R}$ be differentiable with $0 < m \leq f'(x) \leq M$ for $x \in [a, b]$ and let $f(a) < 0 < f(b)$. Given $x_1 \in [a, b]$, define the sequence $\{x_n\}$ by $x_{n+1} = x_n - \dfrac{1}{M} f(x_n)$, for $n \in \mathbb{N}$. Prove that this sequence is well-defined and converges to the unique root $\bar{x}$ of the equation $f(\bar{x}) = 0$ in $[a, b]$. [Hint: Consider $\phi : [a, b] \to \mathbb{R}$ defined by $\phi(x) = x - \dfrac{f(x)}{M}$. Does ϕ map $[a, b]$ into $[a, b] = \text{domain}(f)$?]
Solution: For any $x \in [a, b]$, since $0 < f'(t) \leq M$ for all $t \in [a, b]$, we have $f(x) - f(a) = \displaystyle\int_a^x f'(t)dt \leq M \int_a^x dt = M(x - a)$ which implies that $a = x - \dfrac{f(x)}{M} + \dfrac{f(a)}{M} \leq x - \dfrac{f(x)}{M}$, where we have used the assumption that $f(a) < 0$ in the last inequality.
Similarly, $f(b) - f(x) = \displaystyle\int_x^b f'(t)dt \leq M \int_x^b dt = M(b - x)$ which implies that $b = \dfrac{f(b)}{M} + x - \dfrac{f(x)}{M} + \geq x - \dfrac{f(x)}{M}$, where we have used the assumption that $f(b) > 0$ in the last inequality.
Hence, the function $\phi(x) = x - \dfrac{f(x)}{M}$ maps $[a, b]$ into $[a, b]$, and the sequence $x_{n+1} = x_n - \dfrac{1}{M} f(x_n)$, for $n \in \mathbb{N}$, is well defined if $x_1 \in [a, b]$.
For any $x, y \in [a, b]$, since $|\phi(x) - \phi(y)| = |x - y - \dfrac{f'(c)}{M}(x - y)|$, for some c lying between x, and y.
Hence, $|\phi(x) - \phi(y)| \leq (1 - \dfrac{m}{M})|x - y|$ and ϕ is Lipschitz with Lipschitz constant $0 \leq 1 - \dfrac{m}{M} < 1$.
Hence, ϕ is a contraction and has a unique fixed point $\bar{x} \in [a, b]$.
Since $|x_{n+1} - \bar{x}| = |\phi(x_n) - \phi(\bar{x})| \leq (1 - \dfrac{m}{M})|x_n - \bar{x}| \leq (1 - \dfrac{m}{M})^n |x_1 - \bar{x}| \leq (1 - \dfrac{m}{M})^n (b - a)$,
$\bar{x} = \displaystyle\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n - \lim_{n\to\infty} \frac{f(x_n)}{M} = \bar{x} - \frac{f(\bar{x})}{M}$. Hence, $f(\bar{x}) = 0$.

7. For each integer $k \geq 1$, let $f_k : \mathbb{R} \to \mathbb{R}$ be differentiable satisfying $|f_k'(x)| \leq 1$ for all $x \in \mathbb{R}$, and $f_k(0) = 0$.

(a) For each $x \in \mathbb{R}$, prove that the set $\{f_k(x)\}$ is bounded.
Solution: For each $x \in \mathbb{R}$, since $|f_k(x)| = |f_k(x) - f_k(0)| \leq |f_k'(c_k)\,(x - 0)| \leq |x|$, where c_k lies between x and 0, the set $\{f_k(x)\}_{k=1}^{\infty}$ is bounded.

(b) Use Cantor's diagonal method to show that there is an increasing sequence $n_1 < n_2 < n_3 < \cdots$ of positive integers such that, for every $x \in \mathbb{Q}$, we have $\{f_{n_k}(x)\}$ is a convergent sequence of real numbers.
Solution Let $\mathbb{Q} = \{x_1, x_2, \cdots\}$. Since $\{f_k(x_1)\}$ is bounded, we can extract a convergent subsequence, denoted $\{f_k^1(x_1)\}$, out of $\{f_k(x_1)\}$. Next, the boundedness of $\{f_k^1(x_2)\}$ implies that we can extract a convergent subsequence, denoted by $\{f_k^2(x_2)\}$, out of $\{f_k^1(x_2)\}$. Continuing this way, the boundedness of $\{f_k^j(x_{j+1})\}$ implies that we can extract a convergent subsequence $\{f_k^{j+1}(x_{j+1})\}$, out of $\{f_k^j(x_{j+1})\}$ for each $j \geq 1$. Let $f_{n_k} = f_k^k$ for each $k \geq 1$. Then $\{f_{n_k}\}$ is a subsequence of $\{f_n\}$ and $\{f_{n_k}\}$ converges at each $x_j \in \mathbb{Q}$.

8. Let $\mathscr{F}$ be a bounded and uniformly equicontinuous collection of functions on $D \subset \mathbb{R}^p$ to $\mathbb{R}$ and let f^* be defined on $D \to \mathbb{R}$ by

$$f^*(x) = \sup\{f(x) \mid f \in \mathscr{F}, x \in D\}$$

Show that f^* is continuous on D to $\mathbb{R}$. Show that the conclusion may fail if $\mathscr{F}$ is not uniformly equicontinuous.